

On Physical Measures for Cherry Flows

Liviana Palmisano
Institute of Mathematics of PAN
Warsaw, Poland

October 24, 2014

Abstract

Studies of the physical measures for Cherry flows were initiated in [10]. While the non-positive divergence case was resolved, the positive divergence one still lacked the complete description. Some conjectures were put forward. In this paper we contribute in this direction. Namely, under mild technical assumptions we solve conjectures stated in [10] by providing a description of the physical measures for Cherry flows in the positive divergence case.

1 Introduction

One of the main goals of dynamical systems theory is to describe the typical behavior of orbits, especially when time goes to infinity, and understanding how this behavior is affected by small perturbations of the law that governs the system.

Such questions are especially difficult when the system is sensitive to initial conditions; that is, when a small change in the initial state results in a large variation in the long term behavior of the orbits. One way to address this problem is using the so-called physical measures. These are probability measures of a particular interest as they describe the statistical properties of a large set of orbits.

In general physical measures are still poorly understood. Even their existence has been established only for a narrow class of systems. In this paper we make a contribution in this area by studying the physical measures for Cherry flows.

We recall the classical construction of a Cherry flow given in [2]. It is a \mathcal{C}^∞ flow on the two-dimensional torus without closed orbits and with two singularities, a sink and a saddle. In this case it is relatively easy to check that the only physical measure is the Dirac delta at the sink. The inversed flow, still called a Cherry flow, is a \mathcal{C}^∞ flow on the torus \mathbb{T}^2 , without closed orbits and with two singularities, a saddle point and a repelling point, both hyperbolic. Describing the physical measures in this framework, which is the topic of this paper, is much more difficult and interesting.

Our results answer some open questions of [10].

Before we can state them formally, it is necessary to recall some basic definitions.

1.1 Basic Definitions and General Remarks

Physical Measures. Let ϕ be a continuous flow on a compact manifold M . A probability measure ν is *invariant* under the flow if $\nu(\phi_t(A)) = \nu(A)$, for all $t \in \mathbb{R}$ and for any

measurable set $A \subset M$.

Definition 1.1. Let $t > 0$, we define the following family of probability measures $m_t(z)$ by:

$$\int_M \alpha dm_t(z) = \frac{1}{t} \int_0^t \alpha(\phi_s(z)) ds,$$

for each continuous function $\alpha : M \rightarrow \mathbb{R}$.

Definition 1.2. Let ν be an invariant probability measure. The basin of attraction $B(\nu) = B^\phi(\nu)$ of ν is the set of $z \in M$ such that:

$$\lim_{t \rightarrow \infty} m_t(z) = \nu \text{ (for the weak-} \star \text{ topology)}.$$

The measure ν is said to be physical if its basin of attraction has strictly positive Lebesgue measure.

From now on for the rest of the paper ϕ will denote a Cherry flow on the torus \mathbb{T}^2 , without closed orbits and with two singularities, a saddle point p_s and a repelling point p_r , both hyperbolic.

Let g be the first return map of the flow ϕ to the Poincaré section¹ (it is easy to check that the Poincaré section is equivalent to the unit circle \mathbb{S}^1). In this case g is \mathcal{C}^∞ everywhere except at one point which belongs to the stable manifold of the saddle point and which we will assume to be zero (we identify \mathbb{S}^1 with $[-\frac{1}{2}, \frac{1}{2}]_{-\frac{1}{2} \sim \frac{1}{2}}$). We denote by a and b respectively the left-sided limit and the right-sided limit of the discontinuity point and by $U = (a, b)$.

We consider now the flow φ obtained by reversing the direction of ϕ . The repelling point of ϕ becomes an attractive point for φ which is then a Cherry flow like in Cherry's example ([2]). In this case, the first return map f of φ to the Poincaré section is a circle endomorphism which is \mathcal{C}^∞ everywhere with the exception of the points a and b where it is continuous and it is constant on the interval $U = (a, b)$. Moreover, after a change of coordinates, on a half open neighborhood of these two points f can be written as $x^{\frac{\lambda_1}{-\lambda_2}}$ where $\lambda_1 > 0 > \lambda_2$ are the eigenvalues of the saddle point p_s of ϕ (for more details see [1], [6], [5],[8]).

Rotation Number. As f and g are monotone circle maps, they have a rotation number; this number is the quantity which measures the rate at which an orbit winds around the circle. More precisely, if F is a lifting of f to the real line, the rotation number of f is the limit

$$\rho(f) = \lim_{n \rightarrow \infty} \frac{F^n(x)}{n} \pmod{1}.$$

This limit exists for every x and its value is independent of x . (Obviously we can state the same definition for g).

Fact 1.3. We observe that any Cherry flow φ (respectively ϕ) has a well defined rotation number $\rho \in [0, 1)$ equal to the rotation number of its first return map to any Poincaré section. It is easy to check that the rotation number ρ does not depend on the choice of a Poincaré section.

¹For the construction of the first return map and of the Poincaré section the reader can refer to [6] or [1].

Since the considered flow does not have closed orbits, then f (and g) has an irrational rotation number ρ which admits an expansion as an infinite continued fraction

$$\rho = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\dots}}},$$

where a_i are positive integers.

If we cut off the portion of the continued fraction beyond the n -th position, and write the resulting fraction in lowest terms as $\frac{p_n}{q_n}$ then the numbers q_n for $n \geq 1$ satisfy the recurrence relation

$$q_{n+1} = a_{n+1}q_n + q_{n-1}; \quad q_0 = 1; \quad q_1 = a_1. \quad (1.4)$$

The number q_n is the number of times we have to iterate the rotation by ρ in order that the orbit of any point makes its closest return so far to the point itself (see Chapter I, Sect. I in [3]).

Definition 1.5. Let ρ be the rotation number of φ (respectively ϕ) and let $(a_i)_{i \in \mathbb{N}}$ be the integers of the representation as an infinite continued fraction of ρ . We say that ρ is of bounded type if and only if there exist a positive real number M such that, for any i , $a_i < M$.

First Return Time.

Definition 1.6. Let z be a point of the Poincaré section. The first return time $\tau(z)$ for ϕ to \mathbb{S}^1 is the number of iteration of g needed by z to come back to \mathbb{S}^1 for the first time.

Fact 1.7. The first return time $\tau(z)$ for ϕ to \mathbb{S}^1 has a logarithmic singularity in 0. This means that for all $\epsilon > 0$, there exists a constant $C > 0$ such that, for all z in the interval $(-\epsilon, \epsilon)$, we have that $\frac{1}{C} \leq \frac{\tau(z)}{-\log|z|} \leq C$. In other word $\tau(z)$ is of order of $-\log|z|$.

Since f preserves the order and it does not have periodic points, by Poincaré's Theorem, there exists a continuous order preserving function $h : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ such that $h \circ f = R_\rho \circ h$, where R_ρ is the rotation by ρ . In particular, the order of points in an orbit of f is the same as the order of points in an orbit of R_ρ . Therefore, results about R_ρ can be translated into results about f , via the semi-conjugacy h .

So, the first return function f has only one invariant probability measure μ supported on the minimal set (which is an invariant measure also for g), given by $\mu = h^*(\text{Leb})$. By Proposition 2 in [9] this measure μ can be extended to a invariant probability measure ν on the torus supported on the quasi-minimal set if $\int_{\mathbb{S}^1} \tau d\mu$ is convergent.

1.2 Discussion and Statement of the Results

In this paper we are interested in the physical measures for Cherry flows which are \mathcal{C}^∞ flows on the torus \mathbb{T}^2 , without closed orbits and with two singularities, a saddle point p_s and a repelling point p_r , both hyperbolic.

This problem was studied firstly in [10] in which the authors give a description of the physical measures for some class of Cherry flows. They discover that this problem is related to the variation of the divergence of the flow at the saddle point.²

² Observe that the divergence of a Cherry flow ϕ at the saddle point p_s is exactly the sum of the eigenvalues of the flow ϕ at p_s .

To be more precise, let $\lambda_1 > 0 > \lambda_2$ be the two eigenvalues at the saddle point. In the non-positive divergence case when $\lambda_1 \leq -\lambda_2$, [10] shows that the Dirac deltas at the singularities are the only ergodic invariant probability measures. Moreover [10] establishes that the Dirac delta at the saddle point is the physical measure for the flow.

On the other hand in positive divergence case, $\lambda_1 > -\lambda_2$, in addition to the Dirac deltas at the singularities, there exists another ergodic invariant probability measure ν , which is supported on the quasi-minimal set of the flow. Under additional assumptions of strictly positive divergence, $\lambda_1 > -2\lambda_2$, and of the rotation number of bounded type, they prove that ν is physical and they put forward a conjecture that this holds for any $\lambda_1 > -\lambda_2$.

In this paper we present some new results in this direction.

If the divergence at the saddle point is positive, under the hypothesis that the rotation number is of bounded type, we have:

Theorem 1.8. *Let ϕ be a Cherry flow with eigenvalues at the saddle point $\lambda_1 > 0 > \lambda_2$. If $\lambda_1 > -\lambda_2$ and ϕ has rotation number of bounded type, the ergodic invariant probability measure ν supported on the quasi-minimal set is the physical measure for ϕ with attraction basin having full Lebesgue measure.*

If the divergence at the saddle point becomes strictly positive, *without any assumption* on the rotation number, we have:

Theorem 1.9. *Let ϕ be a Cherry flow with eigenvalues at the saddle point $\lambda_1 > 0 > \lambda_2$. If $\lambda_1 \geq -3\lambda_2$, the ergodic invariant probability measure ν supported on the quasi-minimal set is the physical measure for ϕ with attraction basin having full Lebesgue measure.*

Theorem 1.8 together with results of [10] provides a complete description of the physical measures for Cherry flows having rotation number of bounded type (bounded regime). In the unbounded regime the case $1 < \frac{\lambda_1}{-\lambda_2} < 3$ remains still open. We will comment on technical problems arising in this case in Remark 3.3.

1.3 Standing Assumptions and Notations

Let 0 be the discontinuity point of g . In order to simplify the notation we shall write:

- $\underline{i} = f^i(0)$,
- $\underline{i}_R = R_\rho^i(0)$.

We observe that, because of the properties of f , underlined non-positive integers of the type $\underline{-i}$ represent intervals.

Distance between Points. We denote by $(a, b) = (b, a)$ the shortest open interval between a and b regardless of the order of these two points. The length of that interval in the natural metric on the circle will be denoted by $|a - b|$. Following [4], let us adopt these notational conventions for the distance between the preimages of the first return function f :

- $|\underline{-i}|$ stands for the length of the interval $\underline{-i}$.

- Consider a point x and an interval $\underline{-i}$ not containing it. Then the distance from x to the closest endpoint of $\underline{-i}$ will be denoted by $|(x, \underline{-i})|$, and the distance to the most distant endpoint by $|(x, \underline{-i})|$.
- We define the distance between the endpoints of two intervals $\underline{-i}$ and $\underline{-j}$ analogously. For example, $|(\underline{-i}, \underline{-j})|$ denotes the distance between the closest endpoints of these two intervals while $||\underline{-i}, \underline{-j}||$ stands for $|\underline{-i}| + |(\underline{-i}, \underline{-j})|$.

2 Proof of Theorem 1.8

We consider the sequence

$$\alpha_n = \frac{|(\underline{-q_n}, \underline{0})|}{|[\underline{-q_n}, \underline{0})|}$$

and we prove the following proposition:

Proposition 1. *Let $\lambda_1 > 0 > \lambda_2$ be the eigenvalues of the saddle point of ϕ and let f be the first return function of the reversed flow φ . If f has rotation number of bounded type and $\frac{\lambda_1}{-\lambda_2} \in (1, 2]$, then there exist two constants $K > 0$ and $C < 1$ such that for n big enough, $\frac{-\log \alpha_n}{q_{n+1}} \leq KC^n$. This constant C doesn't depend on the eigenvalues λ_1 and λ_2 of the saddle point.*

Proof. We write $\ell = \frac{\lambda_1}{-\lambda_2}$. Before beginning the proof it is necessary to recall that:

1. by the recursive formula (1.4) $q_0 = 1$, $q_1 = a_1$ and $q_{n+1} = a_{n+1}q_n + q_{n-1}$,
2. by Proposition 6 in [4], for n big enough $\alpha_n \geq K_1 \alpha_{n-1}^{\frac{1-\ell^{-a_{n+1}}}{\ell-1}} \alpha_{n-2}^{\ell^{-a_n}}$ where K_1 is a positive constant.

In order to simplify the proof we will work assuming that, for n big enough,

$$\alpha_n \geq \alpha_{n-1}^{\frac{1-\ell^{-a_{n+1}}}{\ell-1}} \alpha_{n-2}^{\ell^{-a_n}}. \quad (2.1)$$

This hypothesis is not restrictive. Indeed, we observe that for every n we have $q_n \geq \frac{\beta^n}{K_2}$ with $\beta = \frac{1+\sqrt{5}}{2}$ and K_2 a positive constant. Then the estimation remains true also in the general case as we are simply adding to each inequalities below the term $\frac{\log(K_1)}{q_n}$ which trivially satisfies the desired estimation.

To lighten the notation, we introduce a new sequence $(\theta_n)_{n \in \mathbb{N}}$ defined for all n , by $\theta_n := -\log \alpha_n$.

We fix $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, (2.1) is satisfied.

We shall prove the proposition by induction on n . We take $C = C(\ell) = \sup_i \left(\frac{1-\ell^{-a_i}}{(\ell-1)a_i} \right)^{\frac{1}{n_0}}$ and $K \geq \max\{\theta_{n_0-2}, \theta_{n_0-1}\}$. We observe that, for all $1 < \ell \leq 2$, we have $C < 1$; if we consider C as a function of ℓ , then, in the interval $(1, 2]$, $C(\ell)$ is continuous, decreasing and moreover $\lim_{\ell \rightarrow 1} C(\ell) = 1$ and $C(2) < 1$.

We observe that, for any natural number $i \geq 1$

$$\frac{1 - \ell^{-a_i}}{(\ell - 1)a_i} \leq C \quad (2.2)$$

and

$$\ell^{-a_i} \leq \frac{1 - \ell^{-a_i}}{(\ell - 1)a_i} \leq C^{m_0}. \quad (2.3)$$

We now begin the proof by induction.

- Let n_0 be as above. By (2.1) and (2.3), we have that:

$$\begin{aligned} \theta_{n_0} &\leq \left(\frac{1 - \ell^{-a_{n_0+1}}}{\ell - 1} \right) \theta_{n_0-1} + \ell^{-a_{n_0}} \theta_{n_0-2} \\ &\leq C^{m_0} K a_{n_0+1} + C^{m_0} K \\ &\leq K C^{m_0} (a_{n_0+1} + 1). \end{aligned}$$

By point (1) :

$$\theta_{n_0} \leq K C^{m_0} q_{n_0+1}. \quad (2.4)$$

- We now prove the assertion for $n_0 + 1$. By (2.1), (2.4) and (2.3):

$$\begin{aligned} \theta_{n_0+1} &\leq \left(\frac{1 - \ell^{-a_{n_0+2}}}{\ell - 1} \right) \theta_{n_0} + \ell^{-a_{n_0+1}} \theta_{n_0-1} \\ &\leq K C^{m_0} \left(\frac{1 - \ell^{-a_{n_0+2}}}{(\ell - 1)a_{n_0+2}} \right) a_{n_0+2} q_{n_0+1} + K C^{m_0}. \end{aligned}$$

And by point (1) and (2.2) :

$$\begin{aligned} \theta_{n_0+1} &\leq K C^{m_0+1} \left(a_{n_0+2} q_{n_0+1} + \frac{C^{m_0}}{C} \right) \\ &\leq K C^{m_0+1} q_{n_0+2}. \end{aligned}$$

- We now assume that the assertion is true for $n - 2$ and for $n - 1$ and we prove it for n .

By (2.1) and the inductive hypothesis we have that:

$$\begin{aligned} \theta_n &\leq \left(\frac{1 - \ell^{-a_{n+1}}}{\ell - 1} \right) \theta_{n-1} + \ell^{-a_n} \theta_{n-2} \\ &\leq K \left(\frac{(1 - \ell^{-a_{n+1}})}{(\ell - 1)a_{n+1}} C^{m-1} a_{n+1} q_n + C^{m-2} \ell^{-a_n} q_{n-1} \right). \end{aligned}$$

Finally, by (2.2), (2.3) and by point (1)

$$\begin{aligned} \theta_n &\leq K C^n \left(a_{n+1} q_n + \frac{\ell^{-a_n}}{C^2} q_{n-1} \right) \\ &\leq K C^n q_{n+1}. \end{aligned}$$

So, the assertion of the lemma is true for all $n \in \mathbb{N}$ big enough. □

As a direct consequence of Proposition 1 we have the following Corollary:

Corollary 2.5. *Let $\lambda_1 > 0 > \lambda_2$ be the eigenvalues of the saddle point of ϕ and let f be the first return function of the reversing flow φ . If f has rotation number of bounded type and $\frac{\lambda_1}{-\lambda_2} \in (1, 2]$, then there exist two constants $K > 0$ and $C < 1$ such that for n big enough, $\frac{-\log |(q_n, 0)|}{q_{n+1}} \leq KC^n$. This constant C doesn't depend on the eigenvalues λ_1 and λ_2 of the saddle point.*

We recall the following theorem proved in [4] :

Theorem 2.6. *Let $\lambda_1 > 0 > \lambda_2$ be the eigenvalues of the saddle point of ϕ and let f be the first return function of the reversing flow φ . If $\lambda_1 > -\lambda_2$, then $\cup_{i=0}^{\infty} f^{-i}(U)^3$ has full Lebesgue measure on \mathbb{S}^1 .*

The proof of Theorem 1.8 uses the main ideas of the proof of Theorem 3 in [10].

Proof of Theorem 1.8. By Theorem 2 in [10] we know that the flow ϕ has an invariant probability measure ν supported on the quasi-minimal set which corresponds to the extension of the f -invariant measure μ (defined by $\mu = h^*(\text{Leb})$). It remains to prove that ν is a physical measure for ϕ and that its basin of attraction has full Lebesgue measure.

By Theorem 2.6, it is sufficient to prove that the points of the wandering set of φ are in the basin of attraction of ν . Since all points of the wandering set pass through the flat interval of f , then we have just to prove that any point of U is in the basin of attraction of ν .

Let $z \in U$, $\underline{n}_g = g^{n-1}(z)$ and $t_n = \tau(\underline{n}_g)$. For all $t > 0$ there exists $N \in \mathbb{N}$ such that $t = t_1 + t_2 + \dots + t_N + \tilde{t}$ where $0 < \tilde{t} \leq t_{N+1}$. Moreover, let $n \in \mathbb{N}$ such that $q_n \leq N < q_{n+1}$. Since τ is uniformly minored we have that:

$$t \geq CN \quad (2.7)$$

with C a positive constant.

Let m_t be the probability measure on the orbit of flow of length $t > 0$ which starts in z (see Definition 1.1).

Since the only invariant probability measures are δ_s , δ_r and ν (for more details see Theorem 2 in [10]) and since p_r is repelling, then the limits for m_t will be of the form

$$\gamma \delta_s + (1 - \gamma) \nu, \quad (2.8)$$

for some $\gamma \in [0, 1]$. In order to prove that z is in the basin of attraction of ν , which means that $\lim_{t \rightarrow \infty} m_t = \nu$, it is necessary to prove that $\gamma = 0$.

We fix $0 < n_0 < n$ and we prove that the trajectory of z under ϕ spends the most of the time outside of

$$A_{n_0} = \{\phi_s(w) : w \in (\underline{q}_{n_0}, \underline{q}_{n_0+1}), 0 \leq s \leq \tau(w)\}.$$

We prove that, choosing correctly n_0 , the time $t_{A_{n_0}}$ which the trajectory $\phi_s(z)$, $0 \leq s \leq t$ spends in A_{n_0} can be done arbitrarily small in comparison to t , for all t big enough.

In order to do this we divide A_{n_0} and we start to estimate the time t_{B_l} spent by the trajectory $\phi_s(z)$, $0 \leq s \leq t$ in

$$B_l = \{\phi_s(w) : w \in (\underline{q}_l, \underline{q}_{l+2}), 0 \leq s \leq \tau(w)\}.$$

³Observe that $\cup_{i=0}^{\infty} f^{-i}(U)$ is the wandering set of φ .

We observe that, since f is the first return function of the flow obtained reversing the direction of ϕ , if h is the semi-conjugation between f and the rotation R_ρ , then

$$h(\underline{n}_g) = h(g^{n-1}(z)) = h(f^{-n+1}(z)) = h(f^{-n}(0)) = \underline{-n}_R.$$

Then for all $l \in \mathbb{N}$ we have $\underline{q}_l \in (\underline{q}_{l-1}, \underline{q}_{l+1})$ and $\underline{q}_l \in (\underline{q}_{l-1_g}, \underline{q}_{l+1_g})$. So we can say that the number of the points \underline{i}_g , $1 \leq i \leq N$ in $(\underline{q}_l, \underline{q}_{l+2})$ is equal to the number of the points $\underline{-i}_R$, $1 \leq i \leq N$ in $(\underline{q}_l, \underline{q}_{l+2})$.

So we estimate the number N_l of the points $\underline{-i}_R$, $1 \leq i \leq N$ which are in $(\underline{q}_l, 0)$. Since $|(\underline{q}_l, 0)|$ is of the order of $\frac{1}{q_{l+1}}$ and since the rotation is a bijection preserving the distance, we can divide the circle in exactly q_{l+1} disjoint images of $(\underline{q}_l, 0)$ and any image has N_l points $\underline{-i}_R$, $1 \leq i \leq N$.

In conclusion

$$q_{l+1}N_l \leq N$$

and the number of the points $\underline{-i}_R$, $1 \leq i \leq N$ which are in $(\underline{q}_l, \underline{q}_{l+2})$ is less than or equal to $\frac{N}{q_{l+1}}$.

By Fact 1.7, Equation (2.7) and Corollary 2.5⁴ we have that:

$$\begin{aligned} \frac{t_{A_{n_0}}}{t} &= \frac{1}{t} \sum_{l=n_0}^{n-1} t_{B_l} \leq \frac{C_3 N}{t} \sum_{l=n_0}^{n-1} \frac{-\log |(\underline{q}_{l+2}, \underline{0})|}{q_{l+1}} \\ &\leq \frac{C_3}{C} \sum_{l=n_0}^{n-1} \frac{-\log |(\underline{q}_{l+2}, \underline{0})|}{q_{l+1}} \\ &\leq \frac{C_3}{CC_4} \sum_{l=n_0}^{n-1} (C_5)^{l+2}. \end{aligned}$$

Finally, since $\sum_{l=n_0}^{\infty} (C_6)^l$ is convergent, taking n_0 big enough, we can make $\frac{t_{A_{n_0}}}{t}$ as small as we want.

We observe that we have the same result if in place of A_{n_0} we consider A_{n_0-c} with $c > 0$ and $A_{n_0} \Subset A_{n_0-c}$.

It remains to prove that if $\lim_{n_0 \rightarrow \infty} \frac{t_{A_{n_0}}}{t} = 0$ then $\gamma = 0$.

We suppose by contradiction that $\gamma > 0$ and we recall that there exists a sequence strictly non-decreasing sequence of positive reals $(t_n)_{n \in \mathbb{N}}$, $t_n \rightarrow +\infty$ when $n \rightarrow +\infty$, such that: $\lim_{t_n \rightarrow \infty} m_{t_n}(z) = \gamma \delta_s + (1 - \gamma)\nu$ (see (2.8)).

So there exists $T > 0$ such that for all $n \in \mathbb{N}$ for which $t_n > T$ and for any continuous $\alpha : \mathbb{T}^2 \rightarrow \mathbb{R}$

$$\left| \int_{\mathbb{T}^2} \alpha dm_{t_n}(z) - \int_{\mathbb{T}^2} \alpha d(\gamma \delta_s + (1 - \gamma)\nu) \right| < \epsilon. \quad (2.9)$$

Now let $c > 0$ be such that $A_{n_0} \Subset A_{n_0-c}$. Let α be a bump function with compact support such that, for all $x \in A_{n_0}$, $\alpha(x) = 1$ and for all $x \in (A_{n_0-c})^c$, $\alpha(x) = 0$. We observe that

$$\frac{t_{A_{n_0}}}{t_n} \leq \int_{\mathbb{T}^2} \alpha dm_{t_n}(z) \leq \frac{t_{A_{n_0-c}}}{t_n}$$

and we recall that by hypothesis $\lim_{n_0 \rightarrow \infty} \frac{t_{A_{n_0}}}{t_n} = \lim_{n_0 \rightarrow \infty} \frac{t_{A_{n_0-c}}}{t_n} = 0$.

⁴We need a supplementary hypothesis on the eigenvalues $\lambda_1 > 0 > \lambda_2$ of the saddle point. We assume in fact that $\lambda_1 \leq -2\lambda_2$. The case $\lambda_1 > -2\lambda_2$ is proved in [10] (see Theorem 3).

Then, for n big enough, we deduce by (2.9),

$$\gamma - \epsilon < \gamma + (1 - \gamma)\nu(A_{n_0}) - \epsilon < \epsilon \quad (2.10)$$

which is in contradiction with the hypothesis that $\gamma > 0$.

So $\gamma = 0$ and (for the weak- \star topology) $\lim_{t \rightarrow \infty} m_t(z) = \nu$. By Definition 1.2, z is in the basin of attraction of ϕ . \square

3 Proof of Theorem 1.9

The idea of the proof of this theorem is similar to the proof of Theorem 1.8. The main technical tool is the fact that, under the condition $\lambda_1 \geq -3\lambda_2$, without any assumption on the rotation number, the sequence $\frac{|(\underline{0}, \underline{q}_n)|}{|(\underline{0}, \underline{q}_{n-2})|}$ is bounded away from zero (the second claim of Theorem 1.2 in [7]).

Proof of Theorem 1.9. By the second claim of Theorem 1.2 in [7] we can assume that there exist $n_0 \in \mathbb{N}$ and a constant $\alpha \in (0, 1)$ such that $\frac{|(\underline{0}, \underline{q}_n)|}{|(\underline{0}, \underline{q}_{n-2})|} > \alpha^2$ for $n \geq n_0 > 0$. Then, by induction

$$|(\underline{0}, \underline{q}_n)| > C\alpha^n \quad (3.1)$$

for some $C > 0$.

By Theorem 2 in [10], there exists an invariant probability measure ν supported on the quasi-minimal set. We prove that the basin of attraction of ν has full Lebesgue measure; so ν is a physical measure for ϕ .

Like in the proof of Theorem 1.8 we prove that any point of U is in the basin of attraction of ν .

Let $z \in U$, $\underline{n}_g = g^{n-1}(z)$ and $t_n = \tau(\underline{n}_g)$. For all $t > 0$ there exists $N \in \mathbb{N}$ such that $t = t_1 + t_2 + \dots + t_N + \tilde{t}$ where $0 < \tilde{t} \leq t_{N+1}$ and there exists $n \in \mathbb{N}$ such that $q_n \leq N < q_{n+1}$. Since τ is uniformly bounded below, we have that

$$t \geq C_1 N \quad (3.2)$$

with $C_1 > 0$ a positive constant.

Let m_t be the probability measure on the orbit of ϕ of length $t > 0$ which starts at z (see Definition 1.1). The possible limits for m_t must have the form $\gamma\delta_s + (1 - \gamma)\nu$, for some $\gamma \in [0, 1]$. We have to prove that γ is zero (for the details see the proof of Theorem 1.8).

We fix $0 < n_0 < n$ and we prove that the orbit of z under ϕ spends most of time outside of

$$A_{n_0} = \{\phi_s(w) : w \in (\underline{q}_{n_0}, \underline{q}_{n_0+1}), 0 \leq s \leq \tau(w)\}.$$

The time $t_{A_{n_0}}$ spent in A_{n_0} will be calculated as the sum of the times t_{B_i} spent in small pieces of A_{n_0} of the form

$$B_i = \{\phi_s(w) : w \in (\underline{q}_i, \underline{q}_{i+2}), 0 \leq s \leq \tau(w)\}.$$

For these reasons, it is necessary to estimate the number of the points \underline{i}_g , $1 \leq i \leq N$ in $(\underline{q}_i, \underline{q}_{i+2})$ which, like in Theorem 1.8, coincides with the number of the points $\underline{-i}_R$, $1 \leq i \leq N$ in $(\underline{q}_{i_R}, \underline{q}_{i+2_R})$ ⁵ which is less than or equal to $\frac{N}{q_{i+1}}$.

⁵ f is the first return function of ϕ obtained by reversing the direction of ϕ .

Finally by (3.1), (3.2), Fact 1.7 and the fact that $q_l \geq \frac{\beta^l}{C_6}$ for $\beta = \frac{1+\sqrt{5}}{2}$ we have that:

$$\begin{aligned} \frac{t_{A_{n_0}}}{t} &= \frac{1}{t} \sum_{l=n_0}^{n-1} t_{B_l} \leq \frac{C_5 N}{t} \sum_{l=n_0}^{n-1} \frac{-\log |(q_{l+2}, \underline{0})|}{q_{l+1}} \\ &\leq \frac{C_5}{C_1} \sum_{l=n_0}^{n-1} \frac{-\log |(q_{l+2}, \underline{0})|}{q_{l+1}} \\ &\leq \frac{C_5 C \alpha}{C_1} \sum_{l=n_0}^{n-1} \frac{l}{q_{l+1}} \\ &\leq \frac{C_5 C \alpha C_6}{C_1} \sum_{l=n_0}^{n-1} \frac{l}{\beta^{l+1}}. \end{aligned}$$

In conclusion, taking n big enough, we can make $\frac{t_{A_n}}{t}$ as small as we want: then, like in the proof of Theorem 1.8, $\gamma = 0$ and z is in the basin of attraction of ν . \square

Remark 3.3. *The proof of Theorem 1.8 hinges on the recursive estimate (2.1) for α_n (used to demonstrate Proposition 1). In the case of rotation number of bounded type such an estimate was found in [4] and is sufficient to conduct the proof. The case of unbounded type is more problematic as the estimate, provided by [7], is not sufficient any more. To circumvent this problem we assume additionally that the ratio of the eigenvalues at the saddle point is greater than or equal to 3, which assures that the sequence α_n is bounded away from zero (Theorem 1.9). We note that an improved estimate may lead to the complete description of the physical measures for Cherry flows without any assumption on the rotation number.*

Acknowledgments

I would sincerely thank Prof. J. Graczyk for introducing me to the subject of this paper, his valuable advice and continuous encouragement. I am also very grateful to Prof. R. Saghin and Prof. E. Vargas for their willingness to answer to all my questions.

References

- [1] S. Kh. Aranson, G. R. Belitskii, E.V. Zhuzhoma: *Introduction to the Qualitative Theory of Dynamical Systems on Surfaces*. American Mathematical Society, (1996)
- [2] T. Cherry: *Analytic quasi-periodic discontinuous type on a torus*. Proc. Lond. Math. Soc. **44**, 175-215 (1938)
- [3] W. de Melo and S. van Strien: *One-dimensional dynamics*, volume 25 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*. Springer-Verlag, Berlin, (1993)
- [4] J. Graczyk, L. B. Jonker, G. Świątek, F. M. Tangerman and J. J. P. Veerman: *Differentiable Circle Maps with a Flat Interval*. Comm. Math. Phys. **173**, 599-622 (1995)
- [5] M. Martens, S. van Strien, W. de Melo and P. Mendes: *On Cherry flows*. Ergodic Theory and Dynamical Systems **10**, 531-554 (1990)

- [6] I. Nikolaev, E. Zhuzhoma: *Flows on 2-dimensional Manifolds*. Springer- Verlag Berlin Heidelberg, (1999)
- [7] L. Palmisano: *Unbounded regime for circle maps with a flat interval*. To appear in Discrete and Continuous Dynamical System -A
- [8] L. Palmisano: *A phase transition for circle maps and cherry flows*. Comm. Math. Phys. **321**(1), 135-155 (2013)
- [9] R. Saghin, W. Sun, and E. Vargas. *On Dirac physical measures for transitive flows*. Comm. Math. Phys. **298**(3), 741-756 (2010)
- [10] R. Saghin, and E. Vargas. *Invariant measures for Cherry flows*. Comm. Math. Phys. **317**(1), 55-67 (2013)